Stochastic instability of quasi-isolated systems

V. I. Yukalov
Research Center for Optics and Photonics, Instituto de Fisica de São Carlos, Universidade de São Paulo, Caixa Postal 369, São Carlos, São Paulo 13560-970, Brazil
and Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia
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The stability of solutions to evolution equations with respect to small stochastic perturbations is considered. The stability of a stochastic dynamical system is characterized by the local stability index. The limit of this index with respect to infinite time describes the asymptotic stability of a stochastic dynamical system. Another limit of the stability index is given by the vanishing intensity of stochastic perturbations. A dynamical system is stochastically unstable when these two limits do not commute with each other. Several examples illustrate the thesis that there always exist such stochastic perturbations that render a given dynamical system stochastically unstable. The stochastic instability of quasi-isolated systems is responsible for the irreversibility of time arrow.

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I. INTRODUCTION

Evolutional processes of nature are described by differential equations that, in general, are equations in partial derivatives. A set of such partial differential equations constitutes an infinite-dimensional *dynamical system*. Under a *physical system* one implies an ensemble of objects whose behavior is of interest. The evolution of a given physical system is characterized by the related dynamical system. Among physical systems, one distinguishes *isolated systems* as opposed to *open systems*. The evolution of the isolated physical systems is governed by deterministic laws, that is, by *deterministic equations*, not containing random variables. While open physical systems, generally, deal with *stochastic equations*, where random terms represent the interaction with the surroundings.

Solutions to differential equations can be either stable or unstable. There are methods for analyzing the stability of solutions for a given dynamical system, either deterministic [1–3] or stochastic [4]. Here we address another problem, that of stability of a deterministic dynamical system with respect to small stochastic perturbations. This problem is not only interesting by itself but it is of fundamental importance with regard to the question: how adequately the notion of isolated systems represents the physical reality?

As is evident, the notion of an isolated system is an abstraction. In fact, no real system can be completely isolated from its surroundings. This point has been repeatedly emphasized in literature [5–9], and the impossibility of ideally isolating macroscopic systems from their environment is considered as being intimately related with the irreversibility of time [10,11]. Moreover, it has been stressed [12,13] that the concept of an isolated system is logically self-contradictory by its own. This is because to realize the isolation, one has to employ technical devices acting on the system; and to ensure that the latter is kept isolated, one must apply measuring instruments perturbing the system. The preparation and registration processes disturb the system dynamics [14]. In this way, there exists an accepted understanding that any considered physical system is never absolutely isolated but is subject to, probably, weak but, generally, uncontrollable random influence from the environment. Even if this influence is quite weak, its very existence is of principal importance, for explaining the irreversibility of time arrow.

It is worth noting that the irreversible behavior of macroscopic systems is often attributed to internal chaotic nature of microscopic dynamics (see discussion in Ref. [15]). However, not all physical systems display chaotic behavior. Many of them are perfectly governed by rather simple deterministic laws, with no signs of chaos. Nevertheless, the time arrow is well defined for any system, including very simple and nonchaotic ones. Furthermore, the recent developments in dynamical theory, as reviewed by Zaslavsky [16], show that the chaotic dynamics in real systems does not provide finite relaxation time to equilibrium or fast decay of fluctuations, and that chaotic systems are not completely random in the sense originally postulated for statistical systems. Therefore the presence of a random environment, though very weak, seems to be crucially important for interpreting fundamental notions in the behavior of real physical systems.

From another side, there is a common belief, based on practical experience, that physical systems can, with a very good accuracy, be isolated and can be described by deterministic equations, while the random influence of surroundings may be neglected. Thus, there exists an apparent contradiction between the principal necessity of allowing for random perturbations influencing any real system and the practical possibility of neglecting such perturbations, treating a system as isolated.

This contradiction is resolved in the present paper by putting the problem on a firm mathematical footing. The concept of quasi-isolated systems is defined. It is shown that such systems, generally, are unstable with respect to infinitesimally small stochastic perturbations. At the same time, for a finite temporal period, these systems can be treated as approximately isolated.

II. STABILITY OF STOCHASTIC SYSTEMS

Let a continuous variable $x \in \mathbb{D}$ denote a set of spatial coordinates pertaining to a domain $\mathbb{D}$ and let $t \in \mathbb{R}_+$ denote time. Suppose a stochastic field $\xi(t)$ is defined. In general, the latter is a set of stochastic functions $\xi_i(x,t)$, with $i=1,2,\ldots$. Throughout the paper, we shall use the matrix notation [17] making it possible to express the following equations in a compact form. Thus, the *stochastic field* $\xi(t)=[\xi_i(x,t)]$ is considered as a column with respect to both $i=1,2,\ldots$, as well as $x\in\mathbb{D}$. The *dynamical state* $y(\xi,t)=[y_i(x,\xi,t)]$ is also a column with respect to i and x, as is the *velocity field* $v(y,\xi,t)=[v_i(x,y,\xi,t)]$. The set of evolution equations, defining a dynamical system, in the matrix notation reads

$$\frac{d}{dt}y(\xi,t)=v(y,\xi,t). \tag{1}$$

This is complimented by an initial condition

$$y(\xi,0)=y(0), \tag{2}$$

implying the set

$$y_i(x,\xi,0)=y_i(x,0) \quad (i=1,2,\ldots)$$

of the related initial conditions. The averaging over the stochastic field $\xi(t)$ is denoted by the double angular brackets as

$$y(t)=\langle\langle y(\xi,t)\rangle\rangle, \tag{3}$$

which assumes the family of the functions

$$y_i(x,t)=\langle\langle y_i(x,\xi,t)\rangle\rangle, \tag{4}$$

with $i=1,2,\ldots$.

In the stochastic equation (1), the velocity field $v(y,\xi,t)$ may, in general, contain differential as well as integral operations. To solve Eq. (1) means to find the averaged solution (3). Stochastic differential equations, as is known [18], can be defined either in the sense of Ito or in the sense of Stratonovich. In what follows, the latter definition will be employed, which permits simpler calculations and is better motivated physically [19]. It is also possible to use the *stochastic expansion technique* [20,21], presenting the stochastic field as an expansion over smooth functions of spatial and temporal variables with random coefficients. This method enables the usage of the standard differential and integration analysis. The final results of the expansion technique coincide with the corresponding expressions obtained by means of the Stratonovich method.

The local stability of a dynamical system can be characterized by the *local stability index*

$$\sigma(t)\equiv\ln\sup_{\delta y(0)}\frac{|\delta y(t)|}{|\delta y(0)|}, \tag{5}$$

which describes the maximal deviation of the averaged trajectory at time t after an infinitesimal variation of the initial conditions. Such a deviation, according to Eq. (5), corresponds to the law

$$|\delta y(t)|\sim|\delta y(0)|e^{\sigma(t)}, \tag{6}$$

from where it is evident why $\sigma(t)$ is called the stability index, or stability exponent. From this definition, one can immediately conclude that the admissible local properties of motion are classified as

$$\sigma(t)<0 \quad \text{(locally stable)},$$
$$\sigma(t)=0 \quad \text{(locally neutral)}, \tag{7}$$
$$\sigma(t)>0 \quad \text{(locally unstable)}.$$

The asymptotic Lyapunov stability corresponds to the terminology

$$\lim_{t\to\infty}\sigma(t)=-\infty \quad \text{(Lyapunov stable)},$$
$$\lim_{t\to\infty}\sigma(t)>-\infty \quad \text{(Lyapunov unstable)}. \tag{8}$$

And in the language of the Lagrange stability of motion, one has

$$\sup_t\sigma(t)<\infty \quad \text{(Lagrange stable)},$$
$$\sup_t\sigma(t)=\infty \quad \text{(Lagrange unstable)}. \tag{9}$$

The limit

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\sigma(t) \tag{10}$$

corresponds to the largest Lyapunov exponent. One tells that the motion is asymptotically stable if $\lambda<0$, neutral when $\lambda=0$, and unstable if $\lambda>0$.

The usage of a local characteristic of motion, such as the local stability index (5), provides us an essentially richer information on temporal dynamics than the largest Lyapunov exponent (10) defined for the limit $t\to\infty$. First of all, this is because many dynamical systems possess a rather complicated structure of their phase space resembling a topological zoo, consisting of domains of chaotic dynamics as well as of regions of regular motion, containing manifolds of wandering trajectories as well as trapping islands. As a result of this, the fine local properties of orbits play a leading role, while such a fairly rough characteristic as the limiting Lyapunov exponent is less important [16,22].

Moreover, the asymptotic divergence of trajectories of stochastic dynamical systems is not compulsorily exponential [4], because of which making use of only the limiting Lyapunov exponent (10) may result in the loss of information. For example, the divergence of trajectories can be of power law

$$|\delta y(t)|\sim|\delta y(0)|t^{\beta}.$$

Such power laws are typical for weakly disordered systems [23] exhibiting mid-range order [24]. In that case, the local stability index (5) behaves as $\sigma(t)\sim\beta\ln t$, which can be either positive or negative depending on the sign of β. Respectively, the motion is either stable or unstable. While, according to the Lyapunov exponent (10), which is $\lambda=0$, the motion is neutral. Another example has to do with the divergence of trajectories by the stretched exponential law

$$|\delta y(t)| \sim |\delta y(0)| \exp(\kappa t^{\beta}),$$

with $0<\beta<1$, which is also quite ubiquitous in disordered systems. Then the local stability index (5) is $\sigma(t)\sim\kappa t^{\beta}$, which again can be either positive or negative depending on the sign of κ, hence, the motion is either stable or unstable. And the limit (10) is again zero, classifying the motion as neutral.

Instead of the asymptotic Lyapunov exponent (10), one could define the local Lyapunov exponent [25,26] as

$$\lambda(t)=\frac{1}{t}\sigma(t).$$

However, for what follows, the usage of the local stability index (5) is more convenient.

One more advantage of employing a local characteristic of stability is that the limit (10) for many complex systems is technically unachievable. Then the local index (5) is the sole available quantity that can be actually calculated. Such a situation is typical for complicated nonlinear equations that can be treated only numerically [27], for the analysis of various time series that are always finite [28], and for the dynamical representation of perturbation theory, where it is practically feasible to calculate only a finite number of terms [29–32].

The local stability exponent (5) can be expressed through the multiplier matrix $\hat{M}(t)=[M_{ij}(x,x',t)]$ with the elements

$$M_{ij}(x,x',t)\equiv\frac{\delta y_i(x,t)}{\delta y_j(x',0)}. \tag{11}$$

From this definition, it follows that

$$M_{ij}(x,x',0)=\delta_{ij}\delta(x-x'), \tag{12}$$

where δ_{ij} is the Kroneker delta and $\delta(x)$ is the Dirac delta function. Writing the variation of the averaged dynamic state as

$$\delta y(t)=\hat{M}(t)\delta y(0), \tag{13}$$

we see that

$$\sup_{\delta y(0)}\frac{|\hat{M}(t)\delta y(0)|}{|\delta y(0)|}=||\hat{M}(t)||, \tag{14}$$

with the spectral norm of $\hat{M}(t)$ being assumed. Therefore the local stability exponent (5) is

$$\sigma(t)=\ln||\hat{M}(t)||. \tag{15}$$

Thus, to analyze the stability of motion, we need to know the multiplier matrix (11).

III. STOCHASTIC MULTIPLIER MATRIX

What we are actually given is the stochastic equation (1) defining the stochastic dynamic state $y(\xi,t)$, whose variation

$$\delta y(\xi,t)=\hat{M}(\xi,t)\delta y(0) \tag{16}$$

over the initial conditions involves the *stochastic multiplier matrix* $\hat{M}(\xi,t)=[M_{ij}(x,x',\xi,t)]$ with the elements

$$M_{ij}(x,x',\xi,t)\equiv\frac{\delta y_i(x,\xi,t)}{\delta y_j(x',0)}. \tag{17}$$

For the latter, one has the initial condition

$$M_{ij}(x,x',\xi,0)=\delta_{ij}\delta(x-x'). \tag{18}$$

The multiplier matrix (17) is connected with the *stochastic Jacobian matrix* $\hat{J}(\xi,t)=[J_{ij}(x,x',\xi,t)]$ with the elements

$$J_{ij}(x,x',\xi,t)\equiv\frac{\delta v_i(x,y,\xi,t)}{\delta y_j(x',\xi,t)}. \tag{19}$$

The variational differentiation of Eq. (1) gives the equation

$$\frac{d}{dt}\hat{M}(\xi,t)=\hat{J}(\xi,t)\hat{M}(\xi,t) \tag{20}$$

for the multiplier matrix (17). The initial condition for this equation is Eq. (18).

Since the evolution equation (1) represents a set of partial differential equations, one has to define as well boundary conditions. The latter can be written in the general form

$$b(y,\xi,t)=0 \quad (x\in\partial\mathbb{D}), \tag{21}$$

where $\partial\mathbb{D}$ is the boundary manifold of the domain $\mathbb{D}$ and $b(y,\xi,t)=[b_i(x,y,\xi,t)]$ is a boundary vector. Defining the *boundary matrix* $\hat{B}(\xi,t)=[B_{ij}(x,x',\xi,t)]$ with the elements

$$B_{ij}(x,x',\xi,t)\equiv\frac{\delta b_i(x,y,\xi,t)}{\delta y_j(x',\xi,t)} \tag{22}$$

and accomplishing the variation of Eq. (21), we get the boundary condition

$$\hat{B}(\xi,t)\hat{M}(\xi,t)=0 \quad (x\in\partial\mathbb{D}) \tag{23}$$

for the multiplier matrix.

As an illustration, we may offer the often met form of the boundary conditions

$$\left(1+\zeta\frac{\partial}{\partial x}\right)y_i(x,\xi,t)=f_i(t) \quad (x\in\partial\mathbb{D}),$$

where ζ is a parameter and $f_i(t)$ is a given function. The variation of this condition results in the equation

$$\left(1+\zeta\frac{\partial}{\partial x}\right)M_{ij}(x,x',\xi,t)=0 \quad (x\in\partial\mathbb{D}),$$

demonstrating a particular case of the boundary condition (23).

For the multiplier and Jacobian matrices, one may employ different representations. To this end, let a set $\{\varphi_n(t)\}$ of the columns $\varphi_n(t)=[\varphi_{ni}(x,t)]$ be given, forming an orthonormalized complete basis,

$$\varphi_m^+(t)\varphi_n(t)=\delta_{mn}, \quad \sum_n \varphi_n(t)\varphi_n^+(t)=\hat{1},$$

where $\hat{1}=[\delta_{ij}\delta(x-x')]$ is the unity matrix and n is a labeling multi-index. To pass from the x representation to n representation, we define

$$M_{mn}(\xi,t)\equiv\varphi_m^+(t)\hat{M}(\xi,t)\varphi_n(t),$$

$$J_{mn}(\xi,t)\equiv\varphi_m^+(t)\hat{J}(\xi,t)\varphi_n(t). \tag{24}$$

Recall that the matrix notation [17] is used here, according to which, for instance, the action of the multiplier matrix on $\varphi_n(t)$ is the column

$$\hat{M}(\xi,t)\varphi_n(t)=\left[\sum_j \int M_{ij}(x,x',\xi,t)\varphi_{nj}(x',t)dx'\right].$$

Equation (20) for the multiplier matrix in the new representation reads

$$\frac{d}{dt}M_{mn}(\xi,t)=\sum_k \Bigg[J_{mk}(\xi,t)M_{kn}(\xi,t) + M_{mk}(\xi,t)\varphi_k^+(t)\frac{d\varphi_n(t)}{dt} - \varphi_m^+(t)\frac{d\varphi_k(t)}{dt}M_{kn}(\xi,t)\Bigg], \tag{25}$$

where the relation

$$\frac{d\varphi_m^+(t)}{dt}\varphi_n(t)+\varphi_m^+(t)\frac{d\varphi_n(t)}{dt}=0,$$

following from the normalization condition, is used. And from Eq. (18), we have the initial condition

$$M_{mn}(\xi,0)=\delta_{mn} \tag{26}$$

for Eq. (25). The multiplier matrix enjoys several useful properties.

Proposition 1. If the dynamical state $y(\xi,t)$ can be presented as an expansion

$$y(\xi,t)=\sum_n c_n(\xi,t)\varphi_n(t)+f(t), \tag{27}$$

over a basis $\{\varphi_n(t)\}$ and $f(t)=[f_i(x,t)]$ is a column of functions not depending on the initial state $y(0)$, then the multiplier matrix has the form

$$\hat{M}(\xi,t)=\sum_n \mu_n(\xi,t)\varphi_n(t)\varphi_n^+(0), \tag{28}$$

in which

$$\mu_n(\xi,t)\equiv\frac{\delta c_n(\xi,t)}{\delta c_n(\xi,0)}. \tag{29}$$

Proof. The variation of the expansion (27) gives

$$\frac{\delta y_i(x,\xi,t)}{\delta y_j(x',0)}=\sum_n \frac{\delta c_n(\xi,t)}{\delta c_n(\xi,0)}\frac{\delta c_n(\xi,0)}{\delta y_j(x',0)}\varphi_{ni}(x,t).$$

At the same time, from Eq. (27) we have

$$c_n(\xi,t)=\varphi_n^+(t)y(\xi,t)-\varphi_n^+(t)f(t).$$

From the latter equation, we get

$$\frac{\delta c_n(\xi,0)}{\delta y_j(x',0)}=\varphi_{nj}^*(x',0).$$

Using this and invoking the definition (17), we obtain the form (28) with notation (29).

Remarks. Although the basis $\{\varphi_n(t)\}$ is assumed to be orthonormalized, but the vectors $\varphi_m(t_1)$ and $\varphi_n(t_2)$ at different times $t_1\neq t_2$ are not necessarily orthogonal, so that, in general,

$$\varphi_m^+(0)\varphi_n(t)\neq\delta_{mn}.$$

Neither $\varphi_n(t)$ nor $\varphi_n(0)$ are necessarily the eigenvectors of the multiplier matrix, for which we have

$$\hat{M}(\xi,t)\varphi_n(0)=\mu_n(\xi,t)\varphi_n(t).$$

Only when $\varphi_n(t)=\varphi_n$ does not depend on time, then φ_n is an eigenvector of $\hat{M}(\xi,t)$ and $\mu_n(\xi,t)$ is its eigenvalue.

Proposition 2. Suppose the multiplier matrix $\hat{M}(\xi,t)$ possesses eigenvectors $\varphi_n(t)$ forming a complete orthonormalized basis. Then the related eigenvalues, given by the eigenproblem

$$\hat{M}(\xi,t)\varphi_n(t)=\mu_n(\xi,t)\varphi_n(t), \tag{30}$$

can be presented as

$$\mu_n(\xi,t)=\exp\left\{\int_0^t J_{nn}(\xi,t')dt'\right\}. \tag{31}$$

Proof. With $\varphi_n(t)$ being the eigenvectors of the multiplier matrix, the elements of the latter, defined in Eq. (24), are

$$M_{mn}(\xi,t)=\delta_{mn}\mu_n(\xi,t). \tag{32}$$

Substituting this into Eq. (25) yields

$$\delta_{mn}\frac{d}{dt}\mu_n(\xi,t)=J_{mn}(\xi,t)\mu_n(\xi,t)+[\mu_m(\xi,t)-\mu_n(\xi,t)]\varphi_m^+(t)\frac{d\varphi_n(t)}{dt}. \quad (33)$$

When $m=n$, the latter equation gives

$$\frac{d}{dt}\mu_n(\xi,t)=J_{nn}(\xi,t)\mu_n(\xi,t), \quad (34)$$

while for $m\neq n$, it results in

$$J_{mn}(\xi,t)=\left[1-\frac{\mu_m(\xi,t)}{\mu_n(\xi,t)}\right]\varphi_m^+(t)\frac{d\varphi_n(t)}{dt}.$$

Solving Eq. (34), with the initial condition

$$\mu_n(\xi,0)=1, \quad (35)$$

we come to the eigenvalue (31).

Remarks. From the eigenproblem (30), one gets the representation

$$\hat{M}(\xi,t)=\sum_n \mu_n(\xi,t)\varphi_n(t)\varphi_n^+(t) \quad (36)$$

for the multiplier matrix. The eigenvectors of the latter are not necessarily the eigenvectors of the Jacobian matrix (19). Hence the form $J_{mn}(\xi,t)$, defined in Eq. (24), is, in general, nondiagonal.

Proposition 3. Assume that a complete orthonormalized basis $\{\varphi_n(t)\}$ is such that

$$\varphi_m^+(t)\frac{d\varphi_n(t)}{dt}=0 \quad (m\neq n). \quad (37)$$

Then $\varphi_n(t)$ are the eigenvectors of the multiplier matrix $\hat{M}(\xi,t)$ if and only if they are also the eigenvectors of the Jacobian matrix $\hat{J}(\xi,t)$.

Proof. Let condition (37) hold. Then Eq. (25) becomes

$$\frac{d}{dt}M_{mn}(\xi,t)=\sum_k J_{mk}(\xi,t)M_{kn}(\xi,t)+M_{mn}(\xi,t)\times\left[\varphi_n^+(t)\frac{d\varphi_n(t)}{dt}-\varphi_m^+(t)\frac{d\varphi_m(t)}{dt}\right]. \quad (38)$$

If $\varphi_n(t)$ are the eigenvectors of $\hat{M}(\xi,t)$, that is, the form (32) takes place, then Eq. (38) reduces to

$$\delta_{mn}\frac{d}{dt}\mu_n(\xi,t)=J_{mn}(\xi,t)\mu_n(\xi,t),$$

from where it is clear that

$$J_{mn}(\xi,t)=\delta_{mn}J_{nn}(\xi,t). \quad (39)$$

Hence, $\varphi_n(t)$ are the eigenvectors of $\hat{J}(\xi,t)$.

Conversely, if $\varphi_n(t)$ are the eigenvectors of $\hat{J}(\xi,t)$, so that Eq. (39) holds true, then solving Eq. (38) yields

$$M_{mn}(\xi,t)=M_{mn}(\xi,0)\exp\left\{\int_0^t\left[J_{mm}(\xi,t')+\varphi_n^+(t')\frac{d\varphi_n(t')}{dt'}-\varphi_m^+(t')\frac{d\varphi_m(t')}{dt'}\right]dt'\right\}.$$

In view of the initial condition (26), this results in

$$M_{mn}(\xi,t)=\delta_{mn}\exp\left\{\int_0^t J_{nn}(\xi,t')dt'\right\}, \quad (40)$$

which tells us that $\varphi_n(t)$ are the eigenvectors of $\hat{M}(\xi,t)$.

Remarks. As follows from Eq. (40), the eigenvalues of the multiplier matrix are given by expression (31). A simple example, when condition (37) is valid, is the case of a stationary basis $\{\varphi_n\}$, with $\varphi_n(t)=\varphi_n$ not depending on time.

Comparing Eqs. (3), (13), and (16), we see that

$$\hat{M}(t)=\langle\langle\hat{M}(\xi,t)\rangle\rangle. \quad (41)$$

Therefore, if $\hat{M}(\xi,t)$ possesses eigenvectors $\varphi_n(t)$, then the matrix (41) satisfies the eigenproblem

$$\hat{M}(t)\varphi_n(t)=\mu_n(t)\varphi_n(t) \quad (42)$$

with the same eigenvectors and the eigenvalues

$$\mu_n(t)=\langle\langle\mu_n(\xi,t)\rangle\rangle, \quad (43)$$

which have the property

$$\mu_n(0)=1. \quad (44)$$

With the spectral norm

$$||\hat{M}(t)||=\sup_n|\mu_n(t)|,$$

the local stability exponent (15) becomes

$$\sigma(t)=\ln\sup_n|\langle\langle\mu_n(\xi,t)\rangle\rangle|. \quad (45)$$

In this way, the problem of analyzing the stability of a stochastic dynamical system is connected with finding the eigenvalues of the stochastic multiplier matrix.

IV. CONCEPT OF QUASI-ISOLATED SYSTEMS

As is discussed in the Introduction, no real physical system can be completely isolated from its surrounding. The latter can be modeled by stochastic perturbations of the system dynamics. To stress that the amplitude of the stochastic perturbation is small, it is convenient to include explicitly a small factor α in front of the stochastic field $\xi(t)$. So, instead of Eq. (1), we shall write

$$\frac{dy}{dt}=v(y,\alpha\xi,t). \quad (46)$$

The factor $\alpha = \alpha_1 + i\alpha_2$ is assumed to be complex, with its real part $\alpha_1 \equiv \mathrm{Re}\,\alpha$ and imaginary part $\alpha_2 \equiv \mathrm{Im}\,\alpha$. The complex value of the factor α makes it possible to simulate random fluctuations of different physical quantities, such as energy and attenuation or density and phase. If $\alpha \equiv 0$, there are no stochastic perturbations, and one returns to a deterministic dynamical system. When stochastic fields are switched on by means of $\alpha \not\equiv 0$, we have a stochastic dynamical system, whose local stability is characterized by the stability exponent (45) that takes the form

$$\sigma(\alpha,t) \equiv \ln \sup_n |\langle\langle \mu_n(\alpha\xi,t)\rangle\rangle|, \tag{47}$$

where the dependence on the switching factor α is explicitly shown.

The stability exponent (47) describes the stability of a stochastic dynamical system with respect to the infinitesimal variation of initial conditions. For correctly defining the notion of a quasi-isolated system, it is also necessary to consider the stability with respect to infinitesimal stochastic perturbations. This implies that, *after* analyzing the stability of the stochastic system by means of the stability exponent (47), we should set $\alpha \to 0$. Since α is complex valued, the limit $\alpha \to 0$ means that both its real and imaginary parts tend to zero: $\alpha_1 \to 0$ and $\alpha_2 \to 0$. Among all admissible ways of tending to zero for $\alpha \to 0$, it is necessary to choose that one providing the maximal value for the exponent (47), in agreement with its definition as characterizing the *largest* deviation of the trajectory. The so defined limit $\alpha \to 0$ will be denoted as

$$\lim_{\alpha\to 0} \sigma(\alpha,t) \equiv \sup_\alpha \lim_{|\alpha|\to 0} \sigma(\alpha,t). \tag{48}$$

In the stability analysis with the help of the stability exponent (47), an important part is the consideration of the asymptotic stability, when $t \to \infty$. This limit may, in general, not commute with the limit $\alpha \to 0$. Therefore, an important step is to study the commutativity of these limits, characterized by the commutator

$$[\lim_{\alpha\to 0}, \lim_{t\to\infty}] \equiv \lim_{\alpha\to 0}\lim_{t\to\infty} - \lim_{t\to\infty}\lim_{\alpha\to 0}.$$

The content of this section can be summarized by formulating the following definitions.

Definition 1. A physical system is called *quasi-isolated* if its evolution is described by the stochastic dynamical system (46) with infinitesimally small stochastic perturbations.

Definition 2. A quasi-isolated system is *stochastically stable* when

$$[\lim_{\alpha\to 0}, \lim_{t\to\infty}]\sigma(\alpha,t) = 0. \tag{49}$$

Definition 3. A quasi-isolated system is *stochastically unstable* if

$$[\lim_{\alpha\to 0}, \lim_{t\to\infty}]\sigma(\alpha,t) \neq 0. \tag{50}$$

Note that the inclusion of stochastic fields in the evolution equations can be realized in different ways. Hence, in principle, one could consider the stochastic stability with respect to each of the particular ways. A quasi-isolated system may turn out to be stochastically stable with respect to some of perturbations but unstable with respect to others. However, the kind of action of random environment on a quasi-isolated system is, by assumption, unpredictable. Therefore, it is not sufficient to limit ourselves by only some ways of including stochastic perturbations, which would result in the analysis of *partial* stochastic stability. But, in order to make a conclusion on the *general* stochastic stability of a quasi-isolated system, one must analyze all qualitatively different admissible ways of including stochastic terms in the evolution equations. Fortunately, there are just two main qualitatively different types of random noise, multiplicative and additive.

In the following sections, the preceding ideas will be illustrated by concrete examples. Since the perturbing influence of the surroundings may be caused by many independent random sources, their action, according to the central limit theorem, can be modeled by the Gaussian white noise [18]. For the convenience of the reader, the basic properties of this noise, which will be repeatedly used throughout the paper, are listed in short in the Appendix.

V. IMPORTANCE OF MULTIPLICATIVE NOISE

One may notice that additive noise cannot lead to stochastic instability. Really, let the velocity field in Eq. (1) be a sum $v(y,\xi,t) = v_1(y,t) + v_2(\xi,t)$ of two terms, the first of which does not depend on the stochastic field $\xi(t)$, while the second does not include the dynamic state y. Then the Jacobian matrix (19) is defined only through the variation of v_1 and does not depend on v_2. Therefore the solution of Eq. (20) for the multiplier matrix also is independent from v_2, which means that v_2 does not influence the properties of the multiplier matrix, hence, does not change the type of stability.

But the multiplicative noise can strongly influence the stability property. To illustrate this, let us consider the evolution equation (46) with the velocity field

$$v(y,\alpha\xi,t) = f(t) + \alpha\xi(t)y(\alpha\xi,t),$$

where $f(t)$ is a given function and $\xi(t)$ is a Gaussian white-noise variable with the properties described in the Appendix. Equation (46),

$$\frac{dy}{dt} = f(t) + \alpha\xi(t)y, \tag{51}$$

determines the evolution of a one-dimensional dynamical system. In this case, the Jacobian matrix (19) reduces to the function

$$J(\alpha\xi,t) = \alpha\xi(t).$$

According to Eq. (31), this gives the multiplier

$$\mu(\alpha\xi,t) = \exp\left\{\alpha\int_0^t \xi(t')dt'\right\}. \tag{52}$$

The same form (52) could be obtained from the direct variation of the solution

$$y(\alpha\xi,t)=y(0)\exp\left\{\alpha\int_0^t\xi(t')dt'\right\}+\int_0^t f(t')\exp\left\{\alpha\int_{t'}^t\xi(t'')dt''\right\}dt'.$$

For the stability index (47), we find

$$\sigma(\alpha,t)=(\alpha_1^2-\alpha_2^2)\gamma t, \tag{53}$$

where $\alpha_1\equiv\mathrm{Re}\,\alpha$ and $\alpha_2\equiv\mathrm{Im}\,\alpha$. Keeping in mind the definition (48), according to which the stability index is to be maximized with respect to α_1 and α_2, under the given modulus $|\alpha|^2=\alpha_1^2+\alpha_2^2$, we see that $\sup_\alpha(\alpha_1^2-\alpha_2^2)$ equals $|\alpha|^2=\alpha_1^2$. Therefore the index (53) can be written as

$$\sigma(\alpha,t)=|\alpha|^2\gamma t. \tag{54}$$

From here it follows that the limits

$$\lim_{t\to\infty}\lim_{\alpha\to 0}\sigma(\alpha,t)=0, \quad \lim_{\alpha\to 0}\lim_{t\to\infty}\sigma(\alpha,t)=\infty \tag{55}$$

do not commute with each other. This implies that the quasi-isolated system, whose evolution is given by Eq. (51), is stochastically unstable.

VI. OSCILLATOR IN STOCHASTIC BACKGROUND

Many physical processes are presented by oscillatory motion. It is, therefore, illustrative to consider a quasi-isolated system described by a harmonic oscillator subject to the action of a weak external noise. Let the evolution equation (46) have the form

$$\frac{dy}{dt}=i\omega y+\alpha\xi(t)y, \tag{56}$$

where the oscillator frequency ω is real. Here the real part of α corresponds to the noisy attenuation-generation process and the imaginary part of α described the noise of frequency.

For this one-dimensional case, the Jacobian matrix (19) is the function

$$J(\alpha\xi,t)=i\omega+\alpha\,\xi(t).$$

In view of Eq. (31), the multiplier is

$$\mu(\alpha\xi,t)=\exp\left\{i\omega t+\alpha\int_0^t\xi(t')dt'\right\}. \tag{57}$$

The same expression (57) also follows from the variation of the solution

$$y(\alpha\xi,t)=y(0)\exp\left\{i\omega t+\alpha\int_0^t\xi(t')dt'\right\}.$$

The stability index (47) is

$$\sigma(\alpha,t)=(\alpha_1^2-\alpha_2^2)\gamma t, \tag{58}$$

where the properties of the white noise from the Appendix are used.

If the influence of the random noise is removed before the temporal limit, that is, $\alpha\to 0$, then, for any choice of α_1 and α_2, we have

$$\lim_{t\to\infty}\lim_{\alpha\to 0}\sigma(\alpha,t)=0, \tag{59}$$

which corresponds to the neutral motion. However, the situation is different if the limit $t\to\infty$ is taken first. Then, maximizing the factor (58), in agreement with definition (48), as is explained in the preceding section, we get the form (54). As a result,

$$\lim_{\alpha\to 0}\lim_{t\to\infty}\sigma(\alpha,t)=\infty. \tag{60}$$

The noncommutativity of the limits (59) and (60) shows that the oscillatory motion is stochastically unstable.

This means that for a finite time, such that $|\alpha|^2\gamma t\ll 1$, the system with an oscillatory evolution can approximately be treated as isolated. But there always exists such a weak random noise that makes the system unstable for sufficiently long times.

VII. STOCHASTIC DIFFUSION EQUATION

Consider the diffusion equation

$$\frac{\partial y}{\partial t}=[D+\alpha\xi(t)]\frac{\partial^2 y}{\partial x^2}, \tag{61}$$

in which the diffusion constant $D>0$ is subject to weak random fluctuations. For any given finite interval, the spatial variable x can always be scaled so that to be defined on the unity interval. Thus, we assume that $x\in[0,1]$. Equation (61) is complimented by the initial condition

$$y(x,\alpha\xi,0)=y(x,0), \tag{62}$$

with a given function $y(x,0)$, and by the boundary conditions

$$y(0,\alpha\xi,t)=b_0, \quad y(1,\alpha\xi,t)=b_1, \tag{63}$$

where b_0 and b_1 are constant.

For Eq. (61), the Jacobian matrix (19) is

$$J(x,x',\alpha\xi)=[D+\alpha\xi(t)]\frac{\partial^2}{\partial x^2}\delta(x-x'). \tag{64}$$

The boundary conditions (63) lead, according to Eqs. (22) and (23), to the boundary conditions

$$M(0,x',\alpha\xi,t)=M(1,x',\alpha\xi,t)=0 \tag{65}$$

for the multiplier matrix.

Solving the eigenproblem

$$\int_0^1 J(x,x',\alpha\xi)\varphi_n(x')dx' = J_n(\alpha\xi)\varphi_n(x) \qquad (66)$$

for the Jacobian matrix (64), with the boundary conditions

$$\varphi_n(0)=\varphi_n(1)=0, \qquad (67)$$

we find the eigenvalues

$$J_n(\alpha\xi) = -[D+\alpha\xi(t)]k_n^2 \qquad (68)$$

and the eigenfunctions

$$\varphi_n(x)=\sqrt{2}\sin k_n x, \qquad (69)$$

where

$$k_n \equiv \pi n \quad (n=1,2,\ldots,N\to\infty). \qquad (70)$$

The eigenvectors $\varphi_n=[\varphi_n(x)]$, being the columns with the elements (69), are stationary. Hence, they satisfy condition (37). Then, by Theorem 3, the multiplier matrix possesses the same eigenvectors φ_n, with the eigenvalues (31), where $J_{nn}=J_n$. Taking account of Eq. (68) yields

$$\mu_n(\alpha\xi,t)=\exp\left\{-Dk_n^2 t-\alpha k_n^2\int_0^t \xi(t')dt'\right\}. \qquad (71)$$

Note that the solution to Eq. (61) reads

$$y(x,\alpha\xi,t)=\sum_{n=1}^{\infty} c_n\mu_n(\alpha\xi,t)\varphi_n(x)+f(x),$$

where

$$c_n=\int_0^1 [y(x,0)-f(x)]\varphi_n(x)dx, \qquad f(x)=b_0+(b_1-b_0)x.$$

The form of this solution is that of the expansion (27) in Theorem 1, because of which the multiplier matrix could be found by means of this theorem.

Averaging Eq. (71) over the stochastic field (see Appendix), we get

$$|\langle\langle\mu_n(\alpha\xi,t)\rangle\rangle| = \exp(-Dk_n^2 t+\alpha^2 k_n^4\gamma t), \qquad (72)$$

where α is real. Hence, the stability index (47) becomes

$$\sigma(\alpha,t)=\sup_n(-Dk_n^2 t+\alpha^2 k_n^4\gamma t). \qquad (73)$$

Taking into consideration Eq. (70), this gives

$$\sigma(\alpha,t)=\begin{cases} -D\pi^2 t & (\alpha=0),\\ \alpha^2(\pi N)^4\gamma t & (\alpha\neq 0),\end{cases}$$

with $N\to\infty$.

In this way, we have

$$\lim_{t\to\infty}\lim_{\alpha\to 0}\sigma(\alpha,t)=-\infty, \qquad (74)$$

which means that in the absence of any stochastic perturbations the motion would be stable. However, if infinitesimally small stochastic perturbations are present, then

$$\lim_{\alpha\to 0}\lim_{t\to\infty}\sigma(\alpha,t)=\infty, \qquad (75)$$

and the motion is stochastically unstable. This case serves as a good example of how even very weak perturbations can render the system to become unstable, even if without these perturbations it was perfectly stable.

VIII. STOCHASTIC SCHRÖDINGER EQUATION

Consider the nonstationary Schrödinger equation

$$\frac{\partial\psi}{\partial t}=[-iH(\mathbf{r})+\alpha f(\mathbf{r},t)\xi(t)]\psi, \qquad (76)$$

in which we set $\hbar\equiv 1$, $\psi=\psi(\mathbf{r},\alpha\xi,t)$ is a wave function, $H(\mathbf{r})$ is a Hamiltonian, α is real, $f(\mathbf{r},t)$ is a given real function, and $\xi(t)$ is the white noise. With the velocity field defined by the right-hand side of Eq. (76), the Jacobian matrix (19) becomes

$$J(\mathbf{r},\mathbf{r}',\alpha\xi,t)=[-iH(\mathbf{r})+\alpha f(\mathbf{r},t)\xi(t)]\delta(\mathbf{r}-\mathbf{r}'). \quad (77)$$

The eigenproblem for the matrix $\hat{J}(\alpha\xi,t)$, whose elements are given by Eq. (77), reads

$$\hat{J}(\alpha\xi,t)\psi_n=J_n(\alpha\xi,t)\psi_n. \qquad (78)$$

Keeping in mind that α is small, the eigenproblem (78) can be solved by means of perturbation theory. In the zero approximation, the eigenvector $\psi_n=[\psi_n(\mathbf{r})]$ is a column with respect to the spatial variable $\mathbf{r}$, with $\psi_n(\mathbf{r})$ given by the stationary Schrödinger equation

$$H(\mathbf{r})\psi_n(\mathbf{r})=E_n\psi_n(\mathbf{r}),$$

so that the zero-order eigenvalue of the Jacobian matrix is

$$J_n^{(0)}(\alpha\xi,t)=-iE_n.$$

The first-order approximation for the eigenvalue of the Jacobian matrix is given by

$$J_n(\alpha\xi,t)=\psi_n^+\hat{J}(\alpha\xi,t)\psi_n, \qquad (79)$$

which yields

$$J_n(\alpha\xi,t)=-iE_n+\alpha f_n(t)\xi(t), \qquad (80)$$

where

$$f_n(t)\equiv\int \psi_n^*(\mathbf{r})f(\mathbf{r},t)\psi_n(\mathbf{r})d\mathbf{r}.$$

Note that if $f(\mathbf{r},t)=f(t)$ does not depend on the spatial variable $\mathbf{r}$, then the form (80) with $f_n(t)=f(t)$ is an exact eigenvalue of the matrix $\hat{J}(\alpha\xi,t)$. The multi-index n, labeling the eigenvalues, can be discrete as well as continuous.

For the stationary eigenvectors ψ_n of the Jacobian matrix, the multiplier matrix, by Theorem 3, possesses the same eigenvectors and its eigenvalues are

$$\mu_n(\alpha\xi,t)=\exp\left\{-iE_nt+\alpha\int_0^t f_n(t')\xi(t')dt'\right\}. \tag{81}$$

From here, the stability index (47) is

$$\sigma(\alpha,t)=\alpha^2\gamma\int_0^t f_n^2(t')dt'. \tag{82}$$

The function $f(\mathbf{r},t)$ in Eq. (76) can always be chosen so that to satisfy the inequality

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f_n^2(t')dt'>0. \tag{83}$$

Switching off stochastic fields results in the neutral motion, for which

$$\lim_{t\to\infty}\lim_{\alpha\to 0}\sigma(\alpha,t)=0. \tag{84}$$

But for infinitesimally weak stochastic perturbations, the motion becomes unstable, with

$$\lim_{\alpha\to 0}\lim_{t\to\infty}\sigma(\alpha,t)=\infty, \tag{85}$$

where condition (83) is taken into account. In this way, the system described by the Schrödinger equation is stochastically unstable, although for some temporal interval, when $\sigma(\alpha,t)\ll 1$, it can be treated as almost isolated.

IX. SKETCH OF GENERAL SITUATION

In the general case, the stochastic field $\xi(t)=[\xi_i(x,t)]$ is a column composed of the elements $\xi_i(x,t)$ depending on space as well as on time. This field has to enter the evolution equations as a multiplicative noise. To consider a quasi-isolated system, the stochastic term is included with the factor α, which is assumed to be infinitesimally small. For $\alpha \ll 1$, the Jacobian matrix (19) can be calculated by perturbation theory, which yields an expression of the form

$$\hat{J}(\alpha\xi,t)\simeq\hat{J}(0,t)+\alpha\hat{J}'(\xi,t).$$

In the representation of a basis $\{\varphi_n(t)\}$ of vectors $\varphi_n(t)=[\varphi_{ni}(x,t)]$, this reads

$$J_{mn}(\alpha\xi,t)\simeq J_{mn}(0,t)+\alpha\sum_i\int A^i_{mn}(x,t)\xi_i(x,t)dx. \tag{86}$$

If $\varphi_n(t)$ are the eigenvectors of the multiplier matrix $\hat{M}(\alpha\xi,t)$, then its eigenvalues, by Theorem 2, are given by Eq. (31). Averaging these eigenvalues over stochastic fields, implied to be Gaussian, gives

$$\langle\langle\mu_n(\alpha\xi,t)\rangle\rangle=\mu_n(0,t)\exp\left\{\frac{\alpha^2}{2}\kappa_n(t)\right\}, \tag{87}$$

where the factor

$$\mu_n(0,t)=\exp\left\{\int_0^t J_{nn}(0,t')dt'\right\}$$

is the multiplier of an isolated system, without any random perturbations, and

$$\kappa_n(t)=\sum_{ij}\int dx_1\,dx_2\int_0^t A^i_{nn}(x_1,t_1)A^j_{nn}(x_2,t_2)\times\langle\langle\xi_i(x_1,t_1)\xi_j(x_2,t_2)\rangle\rangle dt_1\,dt_2$$

is caused by stochastic perturbations. Then the stability index (47) is

$$\sigma(\alpha,t)=\sup_n \mathrm{Re}\left[\int_0^t J_{nn}(0,t')dt'+\frac{\alpha^2}{2}\kappa_n(t)\right], \tag{88}$$

where α is real.

When $\alpha\to 0$, the stability index

$$\sigma(0,t)=\sup_n \mathrm{Re}\int_0^t J_{nn}(0,t')dt' \tag{89}$$

is defined by the properties of the system without perturbations. The limits $\alpha\to 0$ and $t\to\infty$ do not commute if

$$\lim_{t\to\infty}\left|\frac{\sup_n \mathrm{Re}\,\kappa_n(t)}{\sigma(0,t)}\right|=\infty. \tag{90}$$

Then the quasi-isolated system is stochastically unstable. This condition is accomplished for the concrete cases considered above. It is, of course, impossible to prove that any given quasi-isolated system is, with probability one, stochastically unstable. However, the above consideration suggests, with a high level of probability, that there always exists such a noise that renders stochastically unstable any particular system. This thesis is certainly correct for those systems that, in the absence of noise, display neutral motion. Then $\mathrm{Re}\,J_{nn}(0,t)=0$, hence $\sigma(0,t)=0$, and condition (90) is obviously valid. As is shown in Sec. VII, condition (90) can be held true even for systems that are stable when there is no noise. Let us also emphasize that if, instead of white noise, we would consider infrared noise, then condition (90) would necessarily hold. Really, for a deterministic system at large time, one usually has $\sigma(0,t)\sim t$, while for infrared noise $\kappa_n(t)\sim t^2$. This makes condition (90) evidently valid.

Finally, it is important to note that for stochastic dynamical systems the divergence of averaged trajectories is not necessarily exponential but may be of algebraic form [4], which implies that the norm of the averaged stochastic multiplier matrix has the power-law behavior

$$||\langle\langle\hat{M}(\alpha\xi,t)\rangle\rangle||=\alpha A\left(\frac{t}{t_c}\right)^\beta \quad (\beta>0),$$

where A is a constant and t_c is the *chaotization time* defining the crossover between stable and chaotic motion. For $t \ll t_c$, the motion is stable, while for $t \gg t_c$, it becomes chaotic. The arising instability corresponds to *weak chaos* since the effective trajectory divergence is only algebraic but not exponential. In this case, the local stability index (47) is

$$\sigma(\alpha,t) = \ln\left|\alpha A\left(\frac{t}{t_c}\right)^{\beta}\right|.$$

From here it follows that the limits

$$\lim_{t\to\infty}\lim_{\alpha\to 0}\sigma(\alpha,t) = -\infty, \qquad \lim_{\alpha\to 0}\lim_{t\to\infty}\sigma(\alpha,t) = +\infty$$

do not commute with each other. Therefore such a system is also stochastically unstable.

X. CONCLUSIONS

A convenient characteristic for analyzing the stability of dynamical systems is the *local stability index* (5). This can be expressed through the multiplier matrix $\hat{M}(t)$ as

$$\sigma(t) = \ln||\hat{M}(t)||.$$

For deterministic (nonstochastic) dynamical systems, there exists another representation of the stability index through the Lyapunov or stability matrix $\mathrm{Re}\,\hat{J}(t)$, where $\hat{J}(t)$ is the Jacobian matrix associated with the considered system. The name of the Lyapunov matrix comes from the fact that its eigenvalues are the local Lyapunov exponents. Then the stability index, if condition (37) holds, can be written as

$$\sigma(t) = \int_0^t ||\mathrm{Re}\,\hat{J}(t')||dt'.$$

This presentation, however, is not valid for stochastic dynamical systems. For the latter, the stability index is to be calculated by means of Eq. (45). The form (47) of the stability index,

$$\sigma(\alpha,t) = \ln||\langle\langle \hat{M}(\alpha\xi,t)\rangle\rangle||,$$

is a handy representation for studying the influence of weak stochastic perturbations. The main physical conclusions resulting from the general approach and particular examples are as follows.

(i) *Nonexistence of isolated systems*. The fact that no real physical system can be completely isolated, but is always subject to uncontrollable random perturbations, is more or less generally accepted [5–11]. The point that the concept of an isolated system is logically self-contradictory has also been emphasized [9,12,13]. What is principally important in the present paper is the demonstration that the isolated systems are stochastically unstable with respect to infinitesimally weak random perturbations. A given physical system can be considered as almost isolated, or quasi-isolated, during a finite time interval, but it cannot be treated as such forever. Sooner or later, a quasi-isolated system looses its stability. There are no eternally stable systems in nature.

(ii) *Absence of absolute equilibrium*. In the theory of dynamical systems, solutions are termed equilibrium if they are either constant in time or periodic or quasiperiodic. However, for a quasi-isolated system, none of these solutions can be absolutely stable for an infinitely long time. On a finite temporal interval, a solution can correspond to a stable equilibrium, but with increasing time, some kind of nonequilibrium behavior will certainly appear. For instance, big fluctuations, driving the system far from equilibrium, may arise [33,34]. Since statistical systems are a particular type of real physical systems, they also have to be considered as quasi-isolated. The absence of absolute equilibrium for a statistical system implies that large nonequilibrium fluctuations of mesoscopic scale spontaneously appear in the system, being randomly distributed in space and in time [35]. If evolution equations do possess an attractor, this has to be a chaotic attractor.

(iii) *Irreversibility of time arrow*. As far as completely isolated systems do not exist, but there are only quasi-isolated systems, the dynamics of such a system, because of the action of random perturbations, can never be reversed so that to exactly return to a particular dynamical state. Since quasi-isolated systems are stochastically unstable, any trajectory after sufficiently long time will deviate arbitrarily far from the initial point. All that means the irreversibility of time.

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APPENDIX

Here several formulas, related to the Gaussian white noise, are presented, which have been repeatedly used throughout the paper. The stochastic variable $\xi(t)$, corresponding to this noise, centered at zero, has the properties

$$\langle\langle\xi(t)\rangle\rangle = 0, \quad \langle\langle\xi(t)\xi(t')\rangle\rangle = 2\gamma\delta(t-t'),$$

$$\langle\langle\xi(t_1)\xi(t_2)\cdots\xi(t_{2n+1})\rangle\rangle = 0,$$

$$\langle\langle\xi(t_1)\xi(t_2)\cdots\xi(t_{2n})\rangle\rangle = (2\gamma)^n \sum_{sym}^{(2n-1)!!} \delta(t_1-t_2)\delta(t_3-t_4)\cdots \delta(t_{2n-1}-t_{2n}),$$

where Σ_{sym} implies the symmetrized sum and $(2n-1)!! = (2n)!/2^n n! = 1\times 3\times 5\cdots(2n-1)$. As an example, a symmetrized sum of f_{ij} for $n=2$ means $f_{12}f_{34}+f_{13}f_{24}+f_{14}f_{23}$. The integration of $\xi(t)$ over time gives the Wiener variable

$$w(t) \equiv \int_0^t \xi(t')dt'.$$

For the latter, one has

$$\left\langle\left\langle \int_{t_1}^{t_2} w(t)dw(t) \right\rangle\right\rangle = \gamma(t_2 - t_1),$$

$$\langle\langle w^{2n+1}(t)\rangle\rangle = 0, \quad \langle\langle w^{2n}(t)\rangle\rangle = \frac{(2n)!}{n!}(\gamma t)^n.$$

In general, any Gaussian variable $G(t)$ satisfies the equality

$$\langle\langle \exp G(t)\rangle\rangle = \exp\left\{\frac{1}{2}\langle\langle G^2(t)\rangle\rangle\right\}.$$

For instance,

$$\langle\langle \exp\{\alpha w(t)\}\rangle\rangle = \exp(\alpha^2 \gamma t).$$

These formulas are sufficient to understand all calculations, related to the averaging over the white noise, which have been made in the paper.

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